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The Annihilators of p-Adic Induced Modules

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Let G be a compact p-adic Lie group. We define the Iwasawa Jgebra Λ_G of G, as in [1], to be

$$A, = \varprojlim_{U} \mathbb{Z}_p[G/U],$$

where U runs through the set of open normal subgroups of G.

Let H be a closed subgroup of G. We define the *induced module* to be the left A .-module

$$A_{G/H} = \Lambda_G \otimes_{\Lambda_H} \mathbb{Z}_p,$$

where A, operates trivially on $\mathbb{Z}_p$. The Verma modules introduced in [2, 3] are examples of such modules. We prove the following theorem:

THEOREM. *Suppose* dim $H > \frac{1}{2}$ dim G *(as p-adic Lie groups). Then the two-sided ideal* Ann&,,,) *of elements of* Λ_G *which annihilate all elements of* $A_{G/H}$ *is non-trivial.*

The hypothesis on dimensions is verified, e.g., when G is reductive and the Lie algebra of H is a Bore1 (or, more generally, a minimal parabolic) subalgebra of the Lie algebra of G. It would be interesting to know the structure of $\mathrm{Ann}(\Lambda_{G/H})$ in this case. For example, if G is the principal congruence subgroup $T(p) \subset SL(2, \mathbb{Z}_p)$ and if H is its upper triangular subgroup, is it true that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Ann}(\Lambda_{G/H})$ is a principal ideal in $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A,$? An affirmative answer to this question would be very significant in the study of the arithmetic of elliptic curves over Q.

Proof of the Theorem

It is easy to see that

$$\mathrm{Ann}(\Lambda_{G/H}) = \bigcap_{g \in G/H} \Lambda_G I_{gHg^{-1}}, \tag{1}$$

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where $I_{gHg^{-1}}$ is the kernel of the augmentation map

$$\Lambda_{gHg^{-1}} \to \mathbb{Z}_p, \qquad h \in gHg^{-1} \mapsto 1.$$

We have to show that this intersection is non-trivial.

For any topological space X and any abelian group, we define

$$C^\infty(X, A) = \{\text{locally constant functions } f\colon X \to A\}.$$

Then the Pontryagin dual $\mathrm{Hom}_{\mathrm{cont}}(\Lambda_G, \mathbb{Q}_p/\mathbb{Z}_p)$ (Horn,,,, means continuous homomorphisms) of Λ_G is easily seen to be $C^\infty(G, \mathbb{Q}_p/\mathbb{Z}_p)$ (cf. [2]). This duality becomes a duality of Λ_G-modules if we let

$$g(f(g')) = f(g^{-1}g'), \qquad g, g' \in G, \quad f \in C^\infty(G, \mathbb{Q}_p/\mathbb{Z}_p).$$

Now

$$C^\infty(G, \mathbb{Q}_p/\mathbb{Z}_p) = C^\infty(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p.$$

LEMMA 1. *The module $C^\infty(G, \mathbb{Z})$ is free over $\mathbb{Z}$.*

Proof. We apply the criterion of Lemma 3.5.2.1 of [2]: We have to check that the torsion subgroup of $C^\infty(G, \mathbb{Z})$ is trivial (which is obvious), and that

(*) For every open subgroup $U \subset G$, the group $(C^\infty(G, Z))$” is finitely generated.

In fact, $(C^\infty(G, \mathbb{Z}))^U = C^\infty(G/U, Z)$. Since G/U is finite, (*) is evident: this proves the lemma.

The subgroup $\mathrm{Ann}(\Lambda_{G/H})^\perp$ of $C^\infty(G, \mathbb{Q}_p/\mathbb{Z}_p)$ which is dual to $\mathrm{Ann}(\Lambda_{G/H})$ is the subgroup generated by $\bigcup_{g \in G/H} (\Lambda_g I_{gHg^{-1}})^\perp$, by (1). We have to show that $(\mathrm{Ann}(\Lambda_{G/H}))^\perp \neq C^\infty(G, \mathbb{Q}_p/\mathbb{Z}_p)$. But one checks immediately that

$$\begin{aligned}(\Lambda_G I_{gHg^{-1}})^\perp &= C^\infty(gHg^{-1}\backslash G, \mathbb{Q}_p/\mathbb{Z}_p)\\ &= C^\infty(gHg^{-1}\backslash G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p.\end{aligned} \tag{2}$$

Let M be the subgroup of $C^\infty(G, \mathbb{Z})$ generated by $\bigcup_{g \in G/H} C^\infty(gHg^{-1}\backslash G, Z)$. In view of Lemma 1, we see that

$$\begin{aligned}(\mathrm{Ann}(\Lambda_{G/H}))^\perp = C^\infty(G, \mathbb{Q}_p/\mathbb{Z}_p) &\Leftrightarrow M = C^\infty(G, \mathbb{Z})\\ &\Leftrightarrow M \otimes \mathbb{Q} = C^\infty(G, \mathbb{Q}).\end{aligned}$$

Let d=dimG, $e = \dim H$. We can find a sequence $G \supset G_1 \supset G_2 \cdots \supset$

$G_n \supset \cdots$ of open normal subgroups of G and an integer a such that $\bigcap_n G_n = \{0\}$, and such that, for $n \gg 0$, $[G:G_n] = ap^{dn}$. (For example, imbed G in $GL(m, \mathbb{Z}_p)$ for some $m \gg 0$, and let G_n be the intersection of G with an appropriate principal congruence subgroup of $GL(m, Z,).)$ Let $H_n = H \cap G_n$; then, for some rational number b, we have, for $n \gg 0$, $[H:H_n] = bp^{en}$.

Now

$$\begin{aligned} \dim_{\mathbb{Q}}(C^\infty(gHg^{-1}\backslash G, \mathbb{Q}))^{G_n} &= \dim_{\mathbb{Q}}(gHg^{-1}\backslash G/G_n, \mathbb{Q}) \\ &= [G/G, : gHg^{-1}/gH_n g^{-1}] \\ &= [G/G, : H/H„] \\ &= \frac{a}{b} p^{(d-e)n} \qquad \text{for } n \gg 0. \end{aligned} \tag{3}$$

LEMMA 2. *$(M \otimes \mathbb{Q})^{G_n}$ is generated by*

$$\bigcup_{g \in G/HG_n} C^\infty(gHg^{-1}\backslash G, \mathbb{Q})^{G_n}.$$

Proof. Let $f \in$ (A4 $\otimes \mathbb{Q})^{G_n}$. Then we may write

$$f = \sum_{g \in T} f_g, \qquad \text{where } f_g \in C^\infty(gHg^{-1}\backslash G, \mathbb{Q}) \text{ and } T \text{ is a finite subset of } G/H.$$

Moreover, since all the f_g are locally constant, we may as well assume that, for all g $\in T$, f_g is invariant under G_m for some m $\gg n$. Now consider the map $\mathrm{Tr}_{n/m}: C^\infty(C/G_m, \mathbb{Q}) \to C^\infty(G/G_n, \mathbb{Q})$:

$$(\mathrm{Tr}_{n/m} f)(x) = \sum_{g \in G_n/G_m} f(gx), \qquad x \in G.$$

Then $\mathrm{Tr}_{n/m} f = \sum_{g \in T} \mathrm{Tr}_{n/m} f_g$, where

$$\mathrm{Tr}_{n/m} f_g \in C^\infty(gHg^{-1}\backslash G, \mathbb{Q})^{G_n}.$$

But $\mathrm{Tr}_{n/m} f$ = [G, : $G_m]f$, since $f \in C^\infty(G, \mathbb{Q})^{G_n}$. Thus we may write

$$f = \sum_{g \in T} \frac{1}{[G_n : G_m]} \mathrm{Tr}_{n/m} f_g,$$

which proves the lemma.

It follows that

$$\dim(M \otimes \mathbb{Q})^{G_n} \leqslant \sum_{g \in G/HG_n} \dim C^{\infty}(gHg^{-1} \backslash G, \mathbb{Q})^{G_n}$$

$$= \text{(by (3)) } [G : HG,] \frac{a}{b} p^{(d-e)n} \qquad \text{for } n \gg 0$$

$$= \left(\frac{a}{b}\right)^2 p^{2(d-e)n} \qquad \text{for } n \text{ 4 } 0.$$

But $\dim C^{\infty}(G, \mathbb{Q})^{G_n} = ap^{dn}$ for $n \gg 0$.

Thus, for $n \gg 0$, $\dim(M \otimes \mathbb{Q})^{G_n} < \dim C^{\infty}(G, \mathbb{Q})^{G_n}$, since $2(d-e) < d$ by hypothesis. The theorem is proved.

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